Some properties of p-harmonic maps in homotopy class

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March 16, 2011

Introduction

In the mid 1960's, Eells and Sampson introduced the notion of harmonic map.
Topological relevance (Hartman, Lemaire, Hamilton,...):

- Characterize homotopy of a map $u : M \to N$
- Detect the topology of the involved manifolds

When replacing

$$E(u) = \frac{1}{2}\int |du|^2 \longleftrightarrow E_p(u) = \frac{1}{p}\int |du|^p$$

a natural extension is given by p-harmonic maps (White, Hardt and Lin, Wei,...).

Problem

Which of the well known results holding in the harmonic case can be generalized to the non-linear setting ($p \neq 2$)?

In this talk: some problems related to the existence, uniqueness and triviality of the p-harmonic representative in the homotopy class of a map from a noncompact M to a non-positively curved N.

Let $(M, \langle , \rangle_M)$, $(N, \langle , \rangle_N)$ complete smooth Riemannian manifolds, $\dim M = m, \dim N = n$.

A map $u : M \to N$ is said

- harmonic if stationary point of the energy

$$E(u) = \frac{1}{2} \int_M |du|^2(x) dV_M$$

- p-harmonic if stationary point of the p-energy

$$E_p(u) = \frac{1}{p} \int_M |du|^p(x) dV_M$$

- What is $|du|$?
- What does “stationary” mean?

Let $u : M \to N$, $u \in C^2$, then

$$du : X \in T_x M \mapsto du(X) \in T_{u(x)} N$$

We say that $du \in T^*M \otimes u^{-1}TN$ is a vector-valued 1-form along u.
We define *Hilbert-Schmidt* norm and scalar product on $T^*M \otimes u^{-1}TN$.
Remark. If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then $du \approx J(u)$ and

$$|du|^2_{HS} = \sum_{i=1}^{m} \left(\sum_{A=1}^{n} \left(\frac{\partial u^A}{\partial x^i} \right)^2 \right).$$

In general

$$|du|^2_{HS} = \langle du, du \rangle_{HS} := \mathrm{tr}_M \langle du(\cdot), du(\cdot) \rangle_N$$

Recall: If $\varphi : M \to \mathbb{R}$ a *variation* of φ is a function

$$\varphi_{\xi,t}(x) = \varphi(x) + t\xi(x), \quad t \in [-\epsilon, \epsilon],$$

for some $\xi \in C_c^\infty(M)$ fixed.
For a map $u : M \to N$, let $Z \in u^{-1}TN$ (i.e. $x \in M \mapsto Z(x) \in T_{u(x)}N$) with compact support, and set

$$u_{Z,t}(x) := {}^N\exp_{u(x)}(tZ(x))$$

A map u is said *p-harmonic* if

$$0 = \left.\frac{d}{dt}\right|_{t=0} E_p(u_{Z,t}) = \left.\frac{d}{dt}\right|_{t=0} \frac{1}{p}\int_{\operatorname{supp} Z} |du_{Z,t}|_{HS}^p dV_M$$

for all compactly supported $Z \in u^{-1}TN$.

- **Recall:** $\varphi : M \to \mathbb{R}$ is harmonic $\Leftrightarrow$

$$0 \equiv \Delta\varphi := \operatorname{div} \nabla\varphi = \operatorname{tr}_M \operatorname{Hess} \varphi = \operatorname{tr}_M (Dd\varphi)$$

$\varphi : M \to \mathbb{R}$ is p-harmonic $\Leftrightarrow$

$$0 \equiv \Delta_p\varphi := \operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi)$$

- **Similarly,** $u : M \to N$ is harmonic $\Leftrightarrow$

$$\text{“}\Delta u\text{”} = \tau u := \operatorname{div} du = \operatorname{tr}_M \operatorname{Hess} u \equiv 0$$

- τu is said the *tension field* of u, but
- what do div, Hess mean in this setting?

(a) $-\operatorname{div} := \delta = d^*$ (with respect to the standard L^2 inner product on vector-valued 1-forms)

(b) Introduce a connection

$$D : TM \times \left(T^*M \otimes u^{-1}TN\right) \to T^*M \otimes u^{-1}TN$$
$$(X, \omega) \mapsto D_X\omega,$$

where

$$(D_X\omega)(Y) := {}^N\nabla_{du(X)}(\omega(Y)) - \omega({}^M\nabla_X Y), \quad \forall Y \in TM$$

Then

$$\mathrm{div} := \mathrm{tr}_M D \text{ and } \mathrm{Hess} := Dd$$
$$\tau u := \mathrm{div}\, du = \mathrm{tr}_M \mathrm{Hess}\, u \in u^{-1}TN$$

In fact, with these definitions

$$\left.\frac{d}{dt}\right|_{t=0} E(u_{Z,t}) = \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{supp}\, Z} |du_{Z,t}|^2 dV_M = -\int_M \langle \tau u, Z \rangle_N \, dV_M,$$

so that u is harmonic $\Leftrightarrow \tau u \equiv 0$.

Similarly

$$\frac{d}{dt}\bigg|_{t=0} E_p(u_{Z,t}) = \frac{1}{p}\,\frac{d}{dt}\bigg|_{t=0} \int_{\mathrm{supp}\, Z} |du_{Z,t}|^p dV_M = -\int_M \langle \tau_p u, Z\rangle_N \, dV_M,$$

where

$$\tau_p u = \mathrm{div}\left(|du|^{p-2} du\right) \in u^{-1}TN$$

is the *p-tension field* of u and u is p-harmonic $\Leftrightarrow$ $\tau_p u \equiv 0$.

Remark. In general (e.g. $u \in C^1$) the relation $\tau_p u \equiv 0$ has to be considered in the weak sense, i.e. u is said to be (weakly) p-harmonic if

$$\int_M \left\langle |du|^{p-2} du, d\eta \right\rangle_{HS} = 0$$

$\forall C^\infty$ compactly supported $\eta \in u^{-1}TN$.

Existence and applications: case $\mathbf{p} = 2$

Schoen and Yau studied harmonic maps on noncompact domains.

Theorem (Schoen-Yau, CommMathHelv'76)

Assume M complete, ${}^M\mathrm{Ric} \geq 0$, and N compact, ${}^N\mathrm{Sect} \leq 0$. Let $f : M \to N$ smooth, $|df|^2 \in L^1(M)$. Then f is homotopic to a constant on each compact set.

Strategy of the proof:

(a) an existence result for a (smooth) harmonic map with finite energy in the homotopy class of f

(b) a Liouville type theorem for finite energy harmonic maps.

Pigola-Rigoli-Setti generalized to

$$^{M}\operatorname{Ric} \geq -k(x), \;\; k \geq 0,$$

provided $L = -\Delta - k(x)$ has non-negative spectral radius $\lambda_1(L)$, i.e.

$$\lambda_1(L) := \inf \left\{ \frac{\int |\nabla \varphi|^2 - k(x)\varphi^2}{\int \varphi^2} : \varphi \in C_c^\infty(M) \setminus \{0\} \right\} \geq 0.$$

Theorem (Pigola-Rigoli-Setti, JFA'05)

Assume M complete s.t. $^{M}\operatorname{Ric} \geq -k(x)$, $k(x) \geq 0$, and

$$\lambda_1(-\Delta - k(x)) \geq 0.$$

Let N be compact s.t. $^{N}\operatorname{Sect} \leq 0$. Then, any smooth map $f : M \to N$ of finite energy $|df|^2 \in L^1(M)$ is homotopic to a constant on each compact subset of M.

Existence and applications: case $\mathbf{p} > 2$

Theorem (Wei, Minneapolis'94)

$f : M \to N$, $|df| \in L^p$, N compact with ${}^N\mathrm{Sect} \leq 0$ $\Rightarrow$
$\exists$ $C^{1,\alpha}$ *p-harmonic map u homotopic to f s.t.* $E_p(u) < E_p(f)$.

$\Rightarrow$ we need a vanishing theorem for finite p-energy, p-harmonic maps

Theorem (Pigola-V., GeomDedicata'09)

Let f be a Lipschitz map from a complete M s.t. ${}^M\mathrm{Ric} \geq -k(x)$, $k(x) \geq 0$, *into a compact N s.t.* ${}^N\mathrm{Sect} \leq 0$. *Assume* $|df|^p \in L^1(M)$, $p \geq 2$. *If* ${}^HL = -\Delta - Hk(x)$ *satisfies*

$$\lambda_1\left({}^HL\right) \geq 0,$$

for some

$$H > \begin{cases} p^2/4\,(p-1) & \text{if } p > 2 \\ (m-1)/(m) & \text{if } p = 2, \end{cases}$$

then f is homotopic to a constant.

About the proof. By Wei $\exists u \in C^{1,\alpha}(M,N)$, $|du| \in L^p$, p-harmonic, homotopic to f.

Case $\mathbf{p=2}$. Bochner formula + refined Kato

$$|du|\Delta|du| + k(x)|du|^2 - \frac{1}{m-1}|\nabla|du||^2 \geq 0$$

- The spectral assumption gives $\int |\nabla\varphi|^2 - H\int k(x)\varphi^2 \geq 0, \ \forall\varphi \in C_c^\infty$.
- Integral manipulations with suitable test functions to obtain a Caccioppoli inequality for u

$$\Rightarrow \int_{B(R)} |\nabla|du||^2 \leq O(R^{-2}) \int_M |du|^p$$

- As above

$$\begin{cases} |du| = const. > 0 \\ \lambda_1\left({}^H L\right) \geq 0 \end{cases} \Rightarrow \begin{cases} \mathrm{Vol}(M) < \infty \\ {}^M\mathrm{Ric} \geq 0 \end{cases} \Rightarrow |du| = 0$$

Remarks.

- Our theorem improves also the linear case, by permitting $H < 1$. This weakens both SY and PRS and permits the application to minimal immersions which are δ-stable (in the sense of Colding-Minicozzi, Tam-Zhou...).
- This technique of manipulation first used by Bérard in the field of minimal surfaces. There $|\mathbf{II}|$ (second fundamental form) instead of $|du|$ and Bochner replaced by Simons' inequality

$$|\mathbf{II}|\Delta|\mathbf{II}| + |\mathbf{II}|^4 - \frac{2}{m}|\nabla|\mathbf{II}||^2 \geq 0.$$

Theorem

Every stable oriented minimal hypersurface of $\mathbb{R}^{n+1}$ with finite total curvature is planar.

- [Bérard, PureApplMath'91] proved $n \leq 5$.
- [Shen-Zhu AmerJ'98] $\forall n$ via convergence of minimal surfaces
- Nevertheless the method of Bérard can be adapted to get an abstract proof $\forall n$ [Pigola-V.'10, preprint]

Case p $>$ 2. Same manipuations, but

- The Bochner inequality is

$$|du|\Delta|du| + k(x)|du|^2 \geq -\langle du, d\tau u\rangle$$

where, since u is p-harmonic,

$$\tau u = -(p-2)du(\nabla \log |du|) \neq 0$$

- Now u is not C^∞
- Negative powers of $|du|$ appear

Idea:

- C^1-approximate u on $M_+ = \{|du| > 0\}$ by $u_k \in C^\infty$ (non p-harmonic).
- Prove an L^p-Caccioppoli type inequality fo u_k. The Caccioppoli contains an extra term that vanishes as $k \to \infty$. Take limits to get a Caccioppoli for u.
- Duzaar-Fuchs teach us how to extend this inequality from M_+ to M.

□

Uniqueness

Wei's existence theorem $\Rightarrow$
p-harmonic maps are "canonical" representatives of homotopy classes of maps with finite p-energy $\Rightarrow$
investigate such a space

Problem

How many p-harmonic representatives can be found in a given homotopy class?

First uniqueness result by Wei, [Ind.M.J.'98], for smooth p-harmonic maps defined on compact M (generalizing $p = 2$ due to Hartman).

Aim: detect similar results for complete non-compact manifolds.

Uniqueness: case $p = 2$

Theorem (Schoen-Yau, Topology'79)

Let Vol $M < \infty$*.*

i) *Let* $u : M \to N$ *be harmonic,* $|du| \in L^2$*. If* ${}^N\mathrm{Sect} < 0$*, there's no other harmonic map of finite energy homotopic to* u *unless* $u(M)$ *is contained in a geodesic of* N*.*

ii) *If* ${}^N\mathrm{Sect} \leq 0$ *and* $u, v : M \to N$ *are homotopic harmonic maps with* $|du|, |dv| \in L^2$*, then there is a smooth one-parameter family* $u_t : M \to N$*, of harmonic maps with* $u_0 = u$ *and* $u_1 = v$*. Moreover, for each* $x \in M$*, the curve* $\{u_t(x) : t \in \mathbb{R}\}$ *is a constant (independent of* x*) speed parametrization of a geodesic.*

Remark. If N simply connected $\Rightarrow$ $\mathrm{dist}_N(u, v)$ is constant.

Pigola-Rigoli-Setti [MathZ'08] showed M parabolic is enough (instead of $\operatorname{Vol} M < \infty$).

Definition

M is said *p-parabolic* if every bounded above (weakly) p-subharmonic function is necessarily constant, i.e.

$$\begin{cases} \Delta_p \varphi \geq 0 \ \text{(weakly)} \\ \varphi \leq C < +\infty \end{cases} \quad \Rightarrow \varphi \equiv const.$$

- Several equivalent definitions of p-parabolicity (in term of capacity, Green function...)
- Sufficient conditions in term of volume growth

- We are interested in

Proposition (Kelvin-Nevanlinna-Royden criterion, Gol'dshtein-Troyanov, MathZ'99)

M is p-parabolic, $p > 1$ $\Leftrightarrow$ $\forall X$ vector field s.t.

$$|X| \in L^{\frac{p}{p-1}}(M),\ \operatorname{div} X \in L^1_{loc},\ (\operatorname{div} X)^- := \max\{-\operatorname{div} X; 0\} \in L^1,$$

then $\int_M \operatorname{div} X = 0$.

Remark. $\operatorname{div} X \in L^1_{loc}$ is not necessary. Indeed,

$$\left.\begin{array}{l} M \ p\text{-parabolic} \\ |X| \in L^{\frac{p}{p-1}}(M) \\ \operatorname{div} X \geq f \text{ weakly for some } f \in L^1_{\mathrm{loc}}(M),\ f_- \in L^1(M) \end{array}\right\} \Rightarrow \int_M f \leq 0.$$

Proof (Theorem of SY).

- Let $u, v : M \to N$ be C^∞, harmonic, homotopic s.t. $|du|, |dv| \in L^2(M)$.
- Suppose N is simply connected.
- A direct application of 2nd variation of arc length gives

$$^{N\times N}\operatorname{Hess}\operatorname{dist}_N(Y, Y) \geq 0, \qquad \forall Y \in T(N \times N) \tag{1}$$

  and some necessary conditions for the equality case.
- Then $\operatorname{dist}_N : N \times N \to \mathbb{R}$ is convex.
- $(u, v) : M \to N \times N$ is harmonic
- (convex function)$\circ$(harmonic map) gives (subharmonic function)

- $\Delta\,\mathrm{dist}_N(u,v) \geq 0$, $|du|, |dv| \in L^2$ and parabolicity of $M \Rightarrow$
- $\mathrm{dist}_N(u,v) \equiv const. \Rightarrow$ "=" in (1)
- The necessary conditions for "=" in 2nd var. formula permit to conclude

$\square$

Remark. Schoen and Yau's proof strongly use the property

$$\left.\begin{array}{l} F : M \to N \text{ harmonic} \\ H : N \to \mathbb{R} \text{ convex} \end{array}\right\} \Rightarrow H \circ F \text{ subharmonic}$$

Obstruction to the proof for $p \neq 2$

Problem

$$\left.\begin{array}{l} F : M \to N \ p\textit{-harmonic} \\ H : N \to \mathbb{R} \ \textit{convex} \end{array}\right\} \Rightarrow H \circ F \ p\textit{-subharmonic}?$$

- It's true for p-harmonic morphism (Loubeau [DiffGeomAppl'00]).
- It is folklore that this is not true in general.
- In fact this problem has been pointed out by Lin-Wei among a list of open question in the 2006 Midwest Geometry Conference paper [CommMathAnal'08].

The answer is **NO**. We construct a counter-example.

Let

$$M_g = ([0,+\infty) \times \mathbb{S}^n, ds^2 + g^2(s)d\theta^2)$$
$$N_j = ([0,+\infty) \times \mathbb{S}^n, dt^2 + j^2(t)d\theta^2),$$

where $g, j : [0,+\infty) \to [0,+\infty)$ satisfy

$$g(0) = j(0) = 0, \qquad g'(0) = j'(0) = 1, \qquad g(s), j(t) > 0 \text{ for } s, t > 0.$$

We consider rotationally symmetric map $F : M_g \to N_j$ and function $H : N_j \to \mathbb{R}$, i.e.

$$F(s,\theta) = (f(s), \theta) \qquad \forall s > 0, \theta \in \mathbb{S}^n,$$
$$H(t,\theta) = h(t) \qquad \forall t > 0, \theta \in \mathbb{S}^n,$$

for some functions $f \in C^2([0,\infty),[0,\infty))$ and $h \in C^2([0,\infty),\mathbb{R})$.

Theorem (V., ManuscriptaMath'10)

Suppose that $(n+1) > p > \max\{2, n\}$. *There exist:*

- *warping functions* $g, j : [0, +\infty) \to [0, +\infty)$,
- *a rotationally symmetric p-harmonic map* $F : M_g \to N_j$,
- *a rotationally symmetric convex function* $H : N_j \to \mathbb{R}$,
- *a sequence* $\{s_k\}_{k=1}^{\infty} \Rightarrow +\infty$

such that

$$\Delta_p(H \circ F)(s_k, \theta) < 0.$$

Idea of the proof: using rotationally symmetry conditions, reduce the problem to 1 dimension.

- Cheung-Law-Leung-McLeod [NonlinAnal'98] + our assumption on the manifold $\Rightarrow$ existence result for rotationally symmetric p-harmonic maps.
- Some computations + p-harmonicity of F give

$$\Delta_p(H \circ F) = K\tilde{K}\left\{A_1 + A_2 + A_3\right\},$$

where $K, \tilde{K} > 0$ and

$$A_1 := j'(f)\left[(3-p)(f')^2 + n\frac{j^2(f)}{g^2}\right];$$

$$A_2 := (p-2)j(f)\left[\frac{g'f'}{g} + f''\right];$$

$$A_3 := (p-1)(f')^2 h''(f)\frac{|dF|^2 g^2}{nj(f)h'(f)}.$$

- $\tau_p F = 0$ + assumptions on the warping $\Rightarrow$

$$f'(s) \sim D s^{-\delta}, \qquad \text{as } s \to +\infty,\ D > 0,\ \delta = \delta(n, p, g, j).$$

- By l'Hôpital rule we obtain an estimate for $f''(s_k)$ along a sequence $\{s_k\} \to \infty$
- Along this sequence we can prove that $\Delta_p(H \circ F)(s_k) < 0$.

□

Consequence: one is led to follow different paths in order to deal with the non-linear analogous of Schoen and Yau's result.

Some first steps

Pigola-Rigoli-Setti [MathZ'08]: single map u p-harmonic, homotopic to a constant, $|du|^p \in L^1 \Rightarrow u$ is constant.

Now consider the case $N = \mathbb{R}^n$. If M p-parabolic, then every p-harmonic $u : M \to \mathbb{R}^n$ with $|du| \in L^p(M)$ must be constant. However, using the very special structure of $\mathbb{R}^n$, we are able to extend this conclusion.

Theorem (Holopainen-Pigola-V., PotentialAnalysis)

Suppose that M is p-parabolic, $p > 1$. Let $u, v : M \to \mathbb{R}^n$ be $C^0 \cap W^{1,p}_{loc}(M)$ maps satisfying

$$\begin{cases} \tau_p u = \tau_p v, & \text{if } n > 1, \\ \Delta_p u \geq \Delta_p v & \text{if } n = 1, \end{cases}$$

weakly on M and $|du|, |dv| \in L^p$. Then $u - v$ is constant.

Proof. Assume $o \in M$ s.t. $u(o) = v(o)$. Consider the vector fields X_A defined as

$$X_A := \left[dh_A|_{(u-v)} \circ \left(|du|^{p-2}du - |dv|^{p-2}dv\right)\right]^{\sharp}$$

where $h_A(y) := \sqrt{A + r^2(y)} : \mathbb{R}^n \to \mathbb{R}$, $r(y) = |y|$ and $A > 1$.

Remark 1. $(|du|^{p-2}du - |dv|^{p-2}dv)$ makes sense since $T_u\mathbb{R}^n \sim T_v\mathbb{R}^n$.
2. For $p = 2$, $X_A = \nabla(h_A(u - v)) \Rightarrow \operatorname{div} X_A = \Delta\left(\sqrt{A + |u - v|^2}\right)$

We want to apply Kelvin-Nevanlinna-Royden criterion to X_A

- $|dh_A| \in L^\infty \Rightarrow |X_A|^{\frac{p}{p-1}} \in L^1(M)$.

- $\tau_p u = \tau_p v \Rightarrow$

$$\operatorname{div} X_A = \operatorname{tr}_M \operatorname{Hess} h_A(du - dv, |du|^{p-2} du - |dv|^{p-2} dv)$$

- $\operatorname{Hess} h_A = \frac{\operatorname{Hess} \frac{r^2}{2}}{\sqrt{A+r^2}} - \frac{d\frac{r^2}{2} \otimes d\frac{r^2}{2}}{(A+r^2)^{\frac{3}{2}}}$
- $\operatorname{Hess} \frac{r^2}{2} = \langle , \rangle_{\mathbb{R}^n}$
- A Lindqvist's inequality (true in general vector spaces) gives

$$\left\langle |du|^{p-2} du - |dv|^{p-2} dv, du - dv \right\rangle \geq \Psi(du, dv) = \begin{cases} C_p \left| du - dv \right|^p & p \geq 2 \\ C_p \frac{|du - dv|^2}{(|du| + |dv|)^{2-p}} & 1 < p < 2, \end{cases},$$

Then (here $p > 2$)

$$\operatorname{div} X_A \geq \frac{C_p|du - dv|^p}{\sqrt{A + r^2}} - C_p' r^2 \frac{|du|^p + |dv|^p}{(A + r^2)^{\frac{3}{2}}}$$

$$\Rightarrow (\operatorname{div} X_A)_- \in L^1 \overset{KNR}{\Rightarrow} \int_M \operatorname{div} X_A = 0$$

A limit procedure as $A \to \infty \Rightarrow \int_M |d(u - v)|^p = 0 \Rightarrow u - v \equiv const.$

□

Remark:

- $u, v \in C^0 \cap W^{1,p}_{loc} \Rightarrow$ we need weak computations and the fact that $\forall$ vector field Y on $\mathbb{R}^n$

$$(dr^2)|_{(u-v)(x)}(Y) = 2\,\langle (u - v)(x), Y \rangle_{\mathbb{R}^n}\,.$$

General comparison for $p > 2$

So far:

$p \neq 2$	Pigola-Rigoli-Setti:	• single map homotopic to a constant • general manifold N, ${}^N\mathrm{Sect} \leq 0$
	Holopainen-Pigola-V.:	• two different (non-constant) maps • $N = \mathbb{R}^n$ $\Rightarrow$ trivial topology + good structure
$p = 2$	Schoen-Yau:	• two different (non-constant) maps • general manifold N, ${}^N\mathrm{Sect} \leq 0$

We want to obtain a general comparison (Schoen-Yau's type) for $p > 2$

Theorem (V., JMathAnalAppl)

M, N complete. M p-parabolic, $p \geq 2$.

i) *Let $u : M \to N$ be a $C^{1,\alpha}$ p-harmonic map of finite p-energy. If ${}^N\mathrm{Sect} < 0$, there's no other p-harmonic map of finite p-energy homotopic to u unless $u(M)$ is contained in a geodesic of N.*

ii) *If ${}^N\mathrm{Sect} \leq 0$ and $u, v : M \to N$ are homotopic $C^{1,\alpha}$ p-harmonic maps of finite p-energy, then there is a continuous one-parameter family of maps $u_t : M \to N$ with $u_0 = u$ and $u_1 = v$ such that the p-energy of u_t is constant (indipendent of t) and for each $q \in M$ the curve $t \mapsto u_t(q)$, $t \in [0,1]$, is a constant (indipendent of q) speed parametrization of a geodesic. Moreover, if N is compact, u_t is a p-harmonic maps for each $t \in [0,1]$.*

Remark. If N simply connected $\Rightarrow$ $\mathrm{dist}_N(u, v)$ is constant.

Proof.

- Let $u, v : M \to N$ be $C^{1,\alpha}$, p-harmonic, freely homotopic s.t. $|du|, |dv| \in L^p(M)$.
- Suppose N is simply connected.
- As for $p = 2$ consider $j = (u, v) : M \to N \times N$ and $\operatorname{dist}_N : N \times N \to \mathbb{R}$
- Bad behaviour of composition $\Rightarrow$ define

$$J := (|du|^{p-2}du, |dv|^{p-2}dv) \in T^*M \otimes j^{-1}T(N \times N)$$

along j (note $\operatorname{div} J = 0$) and

$$X_A|_q := \left[dh_A|_{j(q)} \circ J|_q\right]^\sharp, \qquad \text{where } h_A := \sqrt{A + \operatorname{dist}_N^2(\cdot,\cdot)} : N \times N \to \mathbb{R}$$

- We can proceed as in the vector-valued case provided we can estimate ${}^{N\times N}\operatorname{Hess}\operatorname{dist}_N^2(dj, J)$.

Theorem

N simply connected. $X = (X_1, X_2)$ *vector field on* $N \times N$. *Then*

$$\mathrm{Hess}\,\mathrm{dist}_N^2((X_1, X_2), (|X_1|^{p-2}X_1, |X_2|^{p-2}X_2) \geq 0 \qquad (2)$$

+ *some necessary conditions for "="* *in* (2).

- As in the as in the vector-valued case, weak computations + limit procedure $(A \to \infty)$ gives "=" in (2)
- We proceed as for $p = 2$

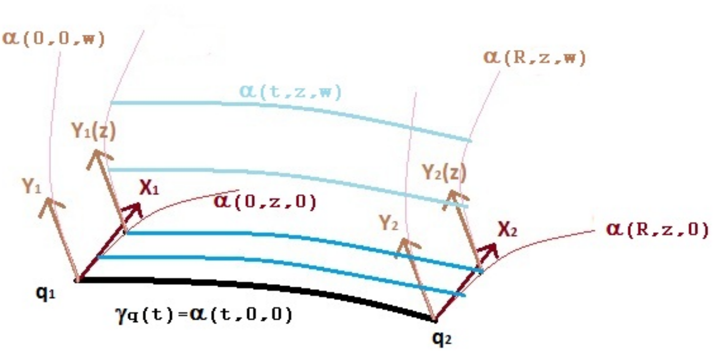
□

Remark. To prove that u_t are p-harmonic, we have no heat flow $\Rightarrow$ we need to assume N is compact (adapting the proof for the existence when $p = 2$ given by Burstall [JLond'84])

About the proof of the ausiliary theorem.

- Let $q = (q_1, q_2) \in N \times N$ and set $Y_i := |X_i|^{p-2} X_i$, $i = 1, 2$.
- Let γ_q be the ! geodesic joining q_1 and q_2 parametrized by arc length.

Idea: set $\bar{R} := \mathrm{dist}_N(q_1, q_2)$ and construct a 2-parameters geodesic variation $\alpha(t, z, w)$ of γ_q, depending on X_i, Y_i,



- Then
$$
{}^{N\times N}\operatorname{Hess}\operatorname{dist}_N(X,Y)=\left.\frac{\partial}{\partial w}\right|_{w=0}\left.\frac{\partial}{\partial z}\right|_{z=0}L(\alpha(\cdot,z,w))
$$
- Using 2nd var. formula for the arc lenght for 2-parameters geodesic variations (Cheeger-Ebin [North-Holland'75])

$$
\begin{aligned}
\left.\frac{\partial^2}{\partial z\partial w}\right|_{z=w=0}L_\alpha(z,w)=&\int_0^{\bar R}\langle\nabla_T Z,\nabla_T W\rangle_N\\
&-\int_0^{\bar R}\left\langle {}^N R(W,T)T,Z\right\rangle_N-\int_0^{\bar R}T\langle Z,T\rangle_N\,T\langle W,T\rangle_N
\end{aligned}
$$

  where Z,W are te Jacobi fields along the 1-parameter geodesic variations $\alpha(t,z,0)$ and $\alpha(t,0,w)$ respectively.
- The explicit form of Z,W is unknown, but using Jacobi equations, the explicit values of Z,W at q_1,q_2 and some integrations by part we can do computations.

□

Perspectives

1. $F : M \to N$ p-harmonic and $H : N \to \mathbb{R}^n$ convex $\overset{\text{in general}}{\not\Rightarrow}$ $\Delta_p(H \circ F) \geq 0$
In the given counterexample

i) the domain manifold M is not p-parabolic and

ii) the target manifold N has positive sectional curvature,

while usually one is in the opposite situation.

Problem

Assume that either M is p-parabolic or ${}^N\mathrm{Sect} \leq 0$ (or both of them). Is $H \circ F$ p-subharmonic?

2. Our general comparison ($p > 2$) requires N is compact to prove that u_t are p-harmonic. If $p = 2$ this assumption is not necessary.
Schoen and Yau's proof makes use of:

- Hamilton's heat flow to solve Dirichlet problem when N is non compact
- Some sort of "maximum principle" for harmonic maps, i.e.

$$\Omega \text{ bounded}, \ \mathsf{dist}_N(u,v)|_{\partial\Omega} \equiv C \Rightarrow \mathsf{dist}_N(u,v) \leq C \text{ on } \Omega$$

Problem

Is the p-laplacian Dirichlet problem for maps from M to N solvable when M is compact with boundary and N is non-compact with ${}^N\mathrm{Sect} \leq 0$*?*

Problem

Are u_t *p-harmonic when* $p > 2$ *and N is non-compact?*